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PG Semester-1
MPHY CC-2 Mathematical Physics

Unit 2 : Elements of complex analysis

Analytic functions

Definition Of Analytic fun^n :- A fun^n $f(z)$ is said to be analytic at a point $z = z_0$ if it is single valued and has a derivative at every point in some neighbourhood of z_0. The fun^n $f(z)$ is said to be analytic in a domain D if it is single valued and differentiable at every point of the domain D. If the term analytic fun^n is used without reference to a specific domain then this term means a fun^n which is analytic in some domain D.

Instead of term analytic the terms regular and holomorphic are also used in the literature of the complex analysis.

Derivatives of an Analytic fun^n

Theorem :- If $f(z)$ is an analytic fun^n in a domain D, then its derivatives of all orders exist in D and they are analytic fun^{n}s in D. The values of these derivatives at any point z_0 in D are given by

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz \quad \text{--①}$$

$$f''(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz \qquad \text{--- (2)}$$

and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi!}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad (n=1,2,3,\ldots) \qquad \text{--- (3)}$$

where C is any closed contour in D surrounding the point $z = z_0$; the contour C being traversed in counterclockwise sense.

For memorizing it may be noted that the values of the derivatives of analytic fun^n $f(z)$ at any point z_0 in D, are obtained formally by repeated differentiation of Cauchy's integral formula equ^n $\int_C f(z)\,dz = 0$ under the integral sign with respect to z_0.

Proof: Let $f(z)$ be analytic within and on a closed contour C and that the points z_0 and $z_0 + \Delta z$ lie within C.

Then by definition the derivative of $f(z)$ at point z_0 is given by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \text{--- (4)}$$

from Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

and $$f(z_0 + \Delta z) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - (z_0 + \Delta z)} dz$$

$$\therefore f'(z_0) = \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \int_C \left(\frac{1}{z - z_0 - \Delta z} - \frac{1}{z - z_0} \right) f(z) dz$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0 - \Delta z)(z - z_0)}$$

But straight-forward calculation shows that

$$\frac{1}{(z - z_0 - \Delta z)(z - z_0)} = \frac{1}{(z - z_0)^2} + \frac{\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz + \lim_{\Delta z \to 0} \frac{\Delta z}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0 - \Delta z)(z - z_0)^2} \quad \text{--- (5)}$$

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Since $f(z)$ is analytic within and on the closed contour C, we can find a positive number M, such that $|f(z)| < M$ and Let L be the length of C.

Then if d is the shortest distance from z_0 to C and if $|\Delta z| < d$, we can write

$$\left| \frac{\Delta z}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0-\Delta z)(z-z_0)^2} \right| < \frac{ML\,|\Delta z|}{2\pi d^2(d-|\Delta z|)}$$

Now it is evident that the right hand side approaches zero when Δz approaches zero, consequently equ^n (5) gives

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2}\,dz \quad \text{—— (1)}$$

let us now proceed to prove formula (2)

By definition

$$f''(z_0) = \lim_{\Delta z \to 0} \frac{f'(z_0+\Delta z) - f'(z_0)}{\Delta z}$$

from (1)

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f'(z_0+\Delta z) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{[z-(z_0+\Delta z)]^2}$$

$$\therefore f'(z_0) = \lim_{\Delta z \to 0} \frac{f'(z_0+\Delta z) - f'(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \int \left[\frac{1}{(z-z_0-\Delta z)^2} - \frac{1}{(z-z_0)^2} \right] f(z)\, dz$$

Straight-forward calculation shows that (6)

$$\frac{1}{\Delta z}\left[\frac{1}{(z-z_0-\Delta z)^2} - \frac{1}{(z-z_0)^2}\right] = \frac{2}{(z-z_0)^3} + \frac{\Delta z\{3(z-z_0) - 2\Delta z\}}{(z-z_0-\Delta z)(z-z_0)^3}$$

Hence equ^n (5) may be written in the form

$$f''(z_0) = \lim_{\Delta z \to 0} \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^2} + \lim_{\Delta z \to 0} \frac{\Delta z}{2\pi i} \int_C \frac{\{3(z-z_0) - 2\Delta z\} f(z)}{(z-z_0-\Delta z)(z-z_0)^3}\, dz \quad \text{———} (7)$$

Since $f(z)$ is analytic, we can find a positive number M' such that

$$|\{3(z-z_0) - 2\Delta z\} f(z)| < M' \text{; hence}$$

$$\left| \frac{\Delta z}{2\pi i} \int_C \frac{\{3(z-z_0) - 2\Delta z\} f(z)}{(z-z_0-\Delta z)(z-z_0)^3}\, dz \right| < \frac{M' L |\Delta z|}{2\pi d^3 (d - |\Delta z|)}$$

Evidently right hand side approaches zero when Δz approaches zero, consequently equ^n (7) gives

$$f''(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}dz \quad \text{— (2)}$$

This there exists the derivative $f'(z)$ at each point z_0 interior to the region bounded by the closed contour C; thereby indicating that $f'(z)$ is analytic at each point within and on the closed contour C.

The argument such in establishing formulas (2) and (3) can be applied successively to obtain a formula for the derivative of any given order. By mathematical induction we obtain the general formula (3) i.e.

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n = 1,2,3 \ldots) \quad \text{— (3)}$$

That is if we assume that this formula is true for any particular integer $n = k$, then by proceeding as before we can show that it is true for $n = k+1$. Thus the requirement that $f(z)$ be analytic not only guarantees a first derivatives of all order

as well, which imply that the derivatives
of $f(z)$ are automatically analytic. This
9 statement assumes the Goursat version of
10 the Cauchy's integral theorem.